MATH 2010F Advanced Calculus I, 2016-17

Solution of Test 2

Q1. Study the following limit. If the limit exist, find the value of the limit. If not, justify your answer with a reason.

(a) (*7 points*)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2},$$

(b) (*8 points*)

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}.$$

Solution.

(a) Take $x_n = \frac{1}{n}$. Then along the sequence $(x_n, 0) \to (0,0)$ as $n \to \infty$, we have

$$\lim_{(x_n,0)\to(0,0)} \frac{x_n \cdot 0}{x_n^2 + 0^2} = 0$$

On the other hand,

$$\lim_{(x_n,x_n)\to(0,0)} \frac{x_n^2}{x_n^2 + x_n^2} = \frac{1}{2}$$

There are two sequences converging to $(0,0)$, so the limit does not exist.

(b) Observe that

$$0 \le \frac{|x^3+y^3|}{x^2+y^2} \le \frac{(|x|+|y|)(x^2+y^2)}{x^2+y^2} = |x|+|y|$$

By using Sandwich rule, we obtain that

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

Consequently,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}\frac{\sin(x^3+y^3)}{x^3+y^3} \\ &= \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^3+y^3} \\ &= 0 \cdot 1 = 0\end{aligned}$$

Q2. Let $f(x,y) = y + \dfrac{x}{y}$.

(a) (*8 points*) Find all the second order partial derivatives in the domain of definition.

(b) (*7 points*) Is f a C^2 function in the domain? Justify your answer with a reason.

Solution.

(a) For $y \neq 0$,

$$f_x = \frac{1}{y}, \quad f_y = 1 - \frac{x}{y^2}$$
$$f_{yx} = f_{xy} = -\frac{1}{y^2}$$
$$f_{xx} = 0, \quad f_{yy} = \frac{2x}{y^3}$$

(b) Since f_{xx}, f_{xy} and f_{yy} are continuous at $y \neq 0$, then f is C^2.

Q3. Let $f(x,y) = \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3$.

(a) (*10 points*) Find $f_x(x,y)$, $f_x(x,y)$ for $(x,y) \neq (0,0)$, and find $f_x(0,0)$ and $f_y(0,0)$.

(b) (*10 points*) Is f differentiable at $(0,0)$? Justify your answer with a reason.

Solution.

(a) Observe that for $(x,y) \neq (0,0)$,

$$f_x(x,y) = 3\left(x^{1/3} + y^{1/3}\right)^2 \cdot \frac{1}{3}x^{-2/3} = \left(x^{1/3} + y^{1/3}\right)^2 x^{-2/3}$$
$$f_y(x,y) = 3\left(x^{1/3} + y^{1/3}\right)^2 \cdot \frac{1}{3}y^{-2/3} = \left(x^{1/3} + y^{1/3}\right)^2 y^{-2/3}$$

By definition,

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 1$$
$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 1$$

(b) If f is differentiable at $(0,0)$ then

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - (x+y)|}{\sqrt{x^2+y^2}} = 0 \tag{1}$$

However (1) does not hold which implies f is not differentiable. Indeed,

$$\lim_{(x,x)\to(0,0)} \frac{|f(x,y) - f(0,0) - (x+y)|}{\sqrt{x^2+y^2}} = \lim_{(x,x)\to(0,0)} \frac{|8x - 2x|}{\sqrt{2x^2}} = \frac{6}{\sqrt{2}} \neq 0$$

Q.4 (*15 points*) Consider the function

$$f(x,y) = \begin{cases} (x^3 + y^4)\cos\left(\frac{1}{x^2+y^2}\right), & \text{for } (x,y) \neq (0,0), \\ 0, & \text{for } (x,y) = (0,0), \end{cases}$$

Show that it is differentiable at $(0,0)$ but its partial derivatives are not continuous there.

Solution. Observe that

$$\begin{aligned} f_x(0,0) &= \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} \\ &= \lim_{x\to 0}\frac{x^3\cos(\frac{1}{x^2})}{x} = 0 \\ f_y(0,0) &= \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} \\ &= \lim_{y\to 0}\frac{y^4\cos(\frac{1}{y^2})}{y} = 0 \end{aligned}$$

Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-0|}{\sqrt{x^2+y^2}} &= \lim_{(x,y)\to(0,0)}\frac{|(x^3+y^4)\cos(\frac{1}{x^2+y^2})|}{\sqrt{x^2+y^2}} \\ &= 0 \end{aligned}$$

since

$$0 \le \frac{|x^3+y^4||\cos(\frac{1}{x^2+y^2})|}{\sqrt{x^2+y^2}} \le \frac{(x^2+|y|^3)(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} = x^2+|y|^3 \to 0$$

Thus, f is differentiable at $(0,0)$.
However, if $(x,y)\neq(0,0)$ then

$$f_x(x,y) = 3x^2\cos(\frac{1}{x^2+y^2}) + \frac{2x(x^3+y^4)}{(x^2+y^2)^2}\sin(\frac{1}{x^2+y^2})$$

When $(x,0)\to(0,0)$,

$$f_x(x,0) = 3x^2\cos(\frac{1}{x^2}) + 2\sin(\frac{1}{x^2}) \nrightarrow 0$$

Therefore, f_x is not continuous at $(0,0)$.
On the other hand, for $(x,y)\neq(0,0)$,

$$f_y(x,y) = 4y^3\cos(\frac{1}{x^2+y^2}) + \frac{2y(x^3+y^4)}{(x^2+y^2)^2}\sin(\frac{1}{x^2+y^2})$$

When $(x,x^{1/2})\to(0,0)$ for $x>0$,

$$f_y(x,x^{1/2}) = 4x^{3/2}\cos(\frac{1}{x^2+x}) + \frac{2(x^{7/2}+x^{5/2})}{(x^2+x)^2}\sin(\frac{1}{x^2+x}) \nrightarrow 0$$

Thus, f_y is also not continuous at $(0,0)$.

Q5. Consider the funtion $g(x,y,z) = z^3 - xy + yz + y^3 - 2$.

(a) (*5 points*) Find its directional derivative along $\xi = (1,2,1)/\sqrt{6}$ at point $P(3,4,7)$,

(b) (*5 points*) Find the direction it increases most rapidly at P,

(c) (*5 points*) Find the direction it decreases most rapidly at P.

Solution.

(a) Observe that

$$\nabla g(x,y,z) = (-y, -x+z+3y^2, 3z^2+y)$$

Then the directional derivative along $\xi = (1,2,1)/\sqrt{6}$ at point $P(3,4,7)$ is

$$\xi \cdot \nabla g(P) = (1,2,1)/\sqrt{6} \cdot (-4, 52, 151) = \frac{251}{\sqrt{6}}$$

(b) The gradient direction is what we want. Indeed,

$$\frac{\nabla g(P)}{|\nabla g(P)|} = \frac{(-4,52,151)}{\sqrt{25521}}$$

(c) g decreases most rapidly at P along its negative gradient direction, i.e.,

$$-\frac{\nabla g(P)}{|\nabla g(P)|} = -\frac{(-4,52,151)}{\sqrt{25521}}$$

Q6. Consider the two dimensional heat equation

$$\partial_t H - \Delta H = 0, \text{ with } \Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}, \quad (x,y) \in \mathbb{R}^2.$$

(a) (*10 points*) Suppose $H(t,x,y) = \frac{1}{t}h(z)$ with $z = \sqrt{\frac{x^2+y^2}{t}}$, show $h(z)$ satisfies the ordinary differential equation

$$h''(z) + \left(\frac{1}{z} + \frac{1}{2}z\right) h'(z) + h(z) = 0,$$

(b) (*10 points*) Can you find all these solution for $h(z)$?

Solution.

(a) By using Chain Rule,

$$\begin{aligned}
(\frac{1}{t}h(z))_x &= \frac{1}{t}h'(z)\frac{x}{\sqrt{t}\sqrt{x^2+y^2}} \\
(\frac{1}{t}h(z))_{xx} &= \frac{1}{t^{3/2}}\left(h''(z)\frac{1}{\sqrt{t}}(\frac{x}{\sqrt{x^2+y^2}})^2 + h'(z)\frac{y^2}{(x^2+y^2)^{3/2}}\right) \\
(\frac{1}{t}h(z))_{yy} &= \frac{1}{t^{3/2}}\left(h''(z)\frac{1}{\sqrt{t}}(\frac{y}{\sqrt{x^2+y^2}})^2 + h'(z)\frac{x^2}{(x^2+y^2)^{3/2}}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta(\frac{1}{t}h(z)) &= \frac{1}{t^{3/2}}\left(\frac{1}{\sqrt{t}}h''(z) + h'(z)\frac{1}{\sqrt{x^2+y^2}}\right) \\
&= \frac{1}{t^2}\left(h''(z) + \frac{1}{z}h'(z)\right)
\end{aligned}$$

And observe that

$$\begin{aligned}
(\frac{1}{t}h(z))_t &= -\frac{1}{t^2}h(z) + \frac{1}{t}h'(z)(-\frac{1}{2}t^{-3/2}\sqrt{x^2+y^2}) \\
&= -\frac{1}{t^2}h(z) - \frac{1}{2t^{5/2}}\sqrt{x^2+y^2}h'(z) \\
&= -\frac{1}{t^2}h(z) - \frac{1}{2t^{5/2}}\sqrt{x^2+y^2}h'(z) \\
&= \frac{1}{t^2}(-h(z) - \frac{1}{2}zh'(z))
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 = \partial_t H - \Delta H &= (\frac{1}{t}h(z))_t - \Delta(\frac{1}{t}h(z)) \\
&= \frac{1}{t^2}(-h(z) - \frac{1}{2}zh'(z)) - \frac{1}{t^2}\left(h''(z) + \frac{1}{z}h'(z)\right)
\end{aligned}$$

which implies

$$0 = h(z) + \frac{1}{2}zh'(z) + h''(z) + \frac{1}{z}h'(z)$$

(b) Observe that

$$\begin{aligned}
& 0 = h''(z) + \left(\frac{1}{z} + \frac{1}{2}z\right)h'(z) + h(z) \\
\Rightarrow \quad & 0 = zh''(z) + \left(1 + \frac{1}{2}z^2\right)h'(z) + zh(z) \\
& \quad = (zh'(z))' + \frac{1}{2}(z^2h(z))'
\end{aligned}$$

Integrating above equation over z from 0 to z on both sides,

$$\begin{aligned}
& zh'(z) + \frac{1}{2}z^2h(z) = C_1 \\
\Rightarrow \quad & h'(z) + \frac{1}{2}zh(z) = C_1\frac{1}{z}
\end{aligned}$$

Using the integrating factor $M = e^{\frac{1}{4}z^2}$, then we have

$$\frac{d}{dz}\left(e^{\frac{1}{4}z^2}h(z)\right) = C_1\frac{1}{z}e^{\frac{1}{4}z^2}$$

Integrating both sides from 0 to z, then we obtain the general solution

$$h(z) = e^{-\frac{1}{4}z^2}\left(C_1\int_0^z \frac{1}{s}e^{\frac{1}{4}s^2}ds + C_2\right)$$

where C_1, C_2 are any constants.